gradient $\operatorname{grad} f(a) = \nabla f(a) =$

$$= \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_d}(a) \right)^T \quad \text{definujeme}$$

pokud ex. $df(a)$ (T.D.).

Derivace podle vektoru v

$$D_v f(a) = \lim_{t \to 0} \frac{f(a + t\cdot v) - f(a)}{t} \in \mathbb{R}$$

Platí:

$$D_v f(a) = \langle \nabla f(a), v \rangle.$$

Derivace ve směru vektoru v: $(v \neq 0)$

$$\frac{\langle \nabla f(a), v \rangle}{\|v\|} = \frac{\|\nabla f(a)\| \cdot \cancel{\|v\|} \cdot \cos\alpha}{\cancel{\|v\|}} = \|\nabla f(a)\| \cdot \cos\alpha$$

Příklad: $f(x,y,z) = x^2 + 2y^2 - z^2$

derivaci ve směru $\overrightarrow{AB}$ v bodě A, kde $A = (-3,2,4)$, $B = (-2,4,2)$.

$$v = B - A = (1,2,-2).$$

$$D_v f(A) = \langle \nabla f(A), (1,2,-2) \rangle =$$

$$\left[\begin{aligned} \nabla f(x,y,z) &= (2x, 4y, -2z) \\ \nabla f(-3,2,4) &= (-6, 8, -8) \end{aligned} \right]$$

$$= \langle (-6,8,-8), (1,2,-2) \rangle = -6 + 16 + 16 = 26$$

Derivace ve směru v je $\frac{26}{\|v\|} = \frac{26}{\sqrt{1+4+4}} = \frac{26}{3}$ //

<u>Úloha</u>: Spočtěte derivaci $f(x,y) = \ln(x^2+y^2)$

$t \mapsto t\cdot \ln t$ $\qquad (x^2+y^2)\cdot \ln(x^2+y^2)$

v bodě (a,b) podle jednotkového vektoru $\underline{\underline{v}}$ kolmého k její vrstevnici směřujícího ven z oblasti ohraničené touto vrstevnicí.

Jak vypadají vrstevnice f: $\{(x,y)\in\mathbb{R}^2 : f(x,y) = \boxed{c}\}$

Nás zajímá bod (a,b), tedy chceme vrstevnici procházející bodem (a,b). $\boxed{c} = ?$

$c = \ln(a^2+b^2)$. Jde o křivku danou rcí:

$$f(x,y) = f(a,b)$$

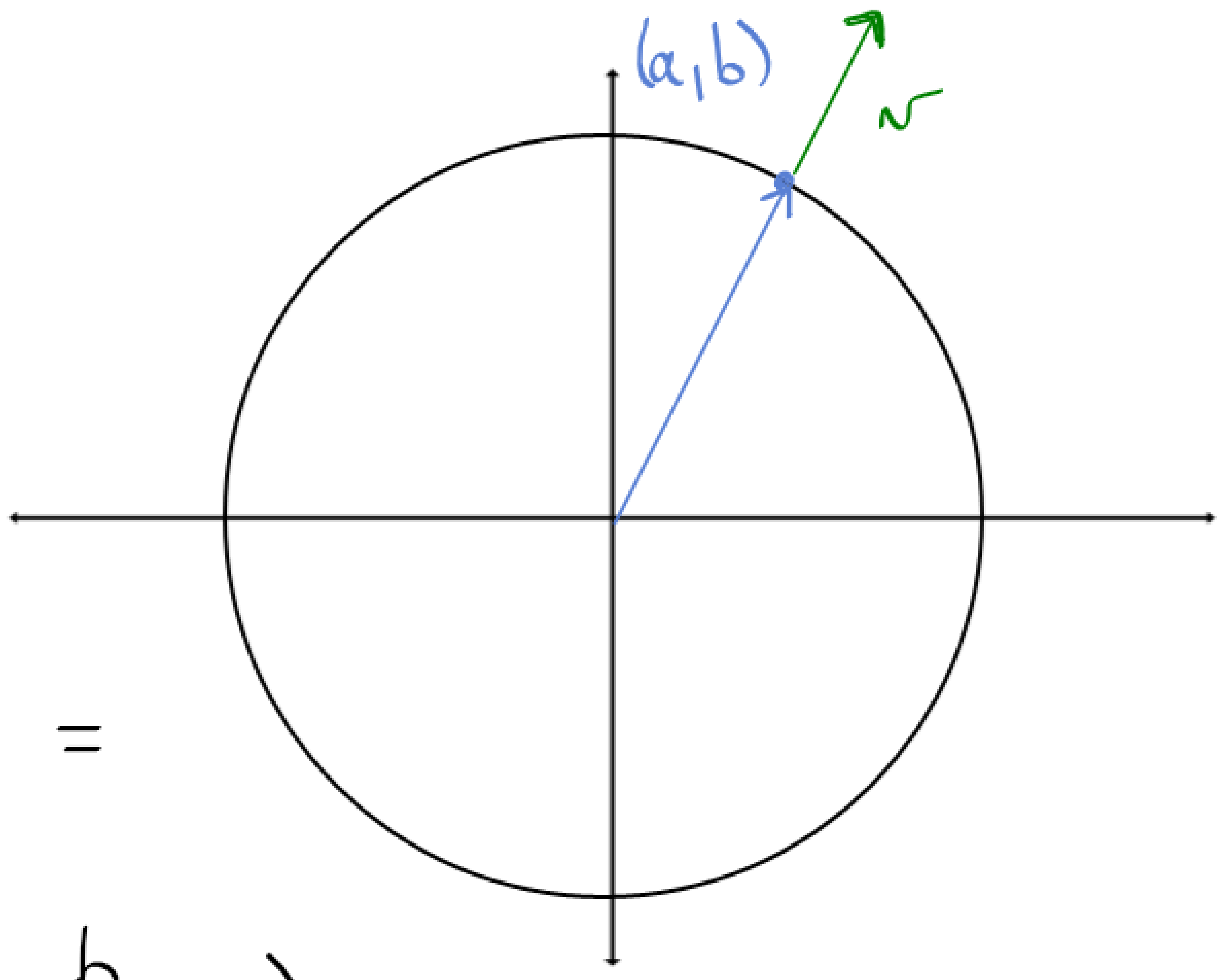
$$\ln(x^2+y^2) = \ln(a^2+b^2)$$

$$x^2+y^2 = \underbrace{\boxed{a^2+b^2}}_{\text{konst.}}$$

Jde o kružnici.

$\|v\| = 1$, „$v \perp \{x^2+y^2 = a^2+b^2\}$“

Je zřejmé, že



$$v = \frac{(a,b)}{\|a,b\|} = \frac{1}{\sqrt{a^2+b^2}}\cdot(a,b) = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

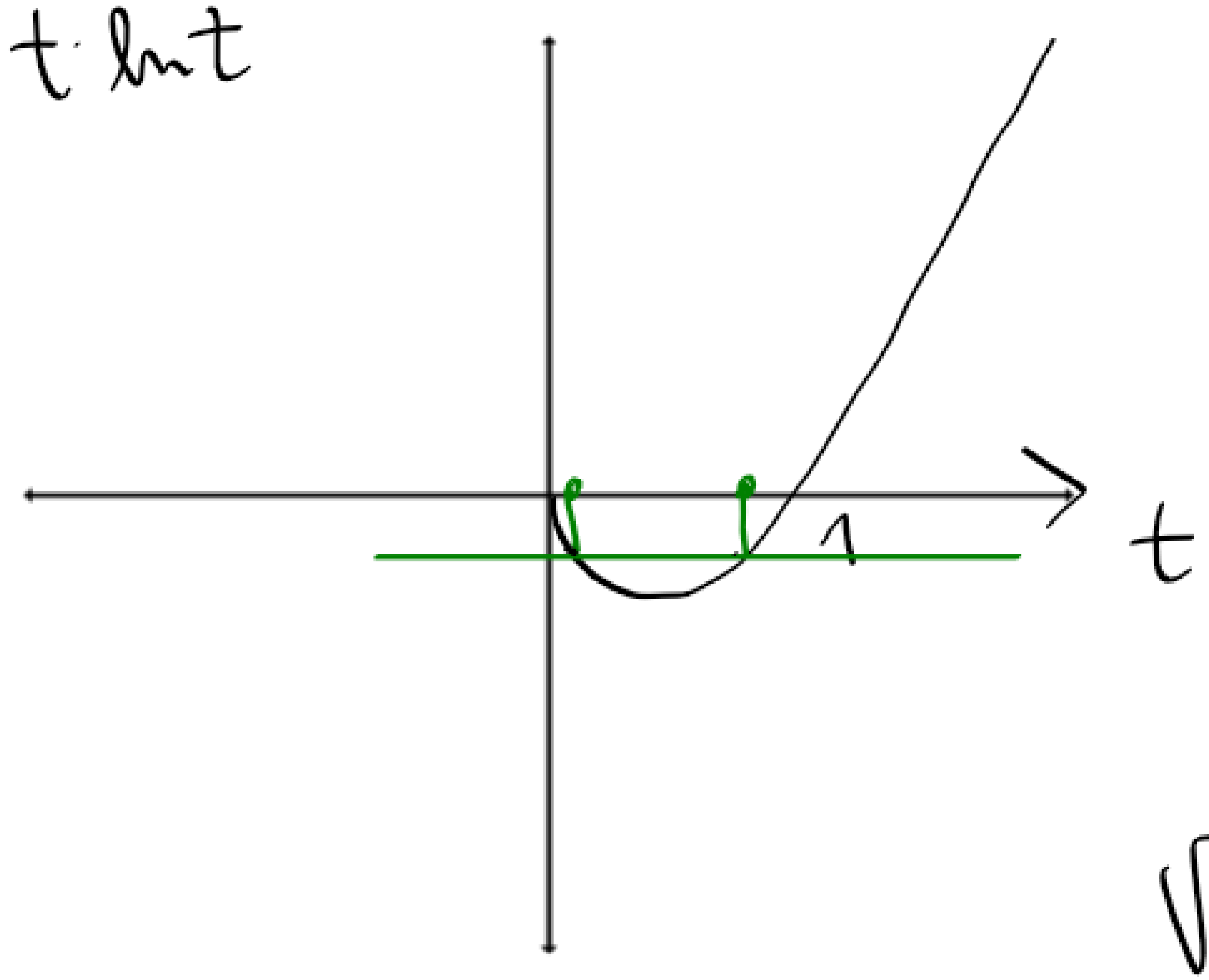
$$D_v f(a,b) = \langle \nabla f(a,b), v \rangle =$$

$$\left[\nabla f(x,y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)\right]$$

$$= \left\langle \left(\frac{2a}{a^2+b^2}, \frac{2b}{a^2+b^2}\right), \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)\right\rangle = (a^2+b^2)^{-3/2}(2a^2+2b^2)$$

$$= 2\cdot(a^2+b^2)^{-1/2}.$$

$t \cdot \ln t$



$$\sqrt{x^2+y^2} \cdot \ln\sqrt{x^2+y^2}$$

$$(x^2+y^2)\ln(x^2+y^2)$$

$$f(x, y, z)$$

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

$$\frac{\partial \tilde f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\tilde f(u,v) = f\big(x(u,v),\ y(u,v),\ z(u,v)\big)$$

$$z = u v^2 w^3 \qquad u = \sin x \qquad v = -\cos x \qquad w = e^x$$

$$\tilde z(x)$$

<u>2 způsoby</u>: (a)

$$\tilde z(x) = \sin x \cdot \cos^2 x \cdot e^{3x} \qquad \tilde z'(x) = \dots \quad \text{(1. SEM.)}$$

(b)
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= v^2 w^3 \cdot \cos x + u \cdot 2v \cdot w^3 \cdot \sin x$$

$$+ u v^2 3 w^2 \cdot e^x =$$

$$= \cos^2 x\, e^{3x} \cos x + 2\sin x\,(-\cos x)\, e^{3x} \cdot \sin x +$$

$$+ \sin x \cos^2 x\, 3 e^{2x} \cdot e^x.$$

$f(x,y) = x^2 + 3y^2$

určete derivace f vzhledem k r, α,

$x = r \cdot \cos\alpha$, $y = r \cdot \sin\alpha$

$$\tilde{f}(r,\alpha) = f\big(x(r,\alpha), y(r,\alpha)\big) = \\ = f(r\cdot\cos\alpha, r\cdot\sin\alpha)$$

$$\frac{\partial \tilde{f}}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= 2x \cdot \cos\alpha + 6y \cdot \sin\alpha =$$

$$= 2r \cdot \cos^2\alpha + 6r \cdot \sin^2\alpha =$$

$$= 2r + 4r\sin^2\alpha$$

$$\frac{\partial \tilde{f}}{\partial \alpha} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} =$$

$$= 2x \cdot (-r\sin\alpha) + 6y \cdot (r\cos\alpha) =$$

$$= -2r^2 \sin\alpha\cos\alpha + 6r^2 \sin\alpha\cos\alpha =$$

$$= \sin\alpha\cos\alpha \cdot 4r^2 = 2r^2 \sin 2\alpha$$

$g(x,y) = f(x+y,\ x-y)$ $\qquad \frac{\partial^2 g}{\partial x \partial y}(a,b)$

" $f = f(u,v)$ "

$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial u}\cdot\frac{\partial(x+y)}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial(x-y)}{\partial y} =$$

$$= \frac{\partial f}{\partial u}(x+y,x-y)\cdot 1 + \frac{\partial f}{\partial v}(x+y,x-y)\cdot(-1)$$

$$\frac{\partial^2 g(x,y)}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}(x+y,x-y) - \frac{\partial f}{\partial v}(x+y,x-y)\right)$$

$$= \frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial(x+y)}{\partial x} + \frac{\partial^2 f}{\partial v\partial u}\cdot\frac{\partial(x-y)}{\partial x} -$$

$$-\left(\frac{\partial^2 f}{\partial u\partial v}\cdot\frac{\partial(x+y)}{\partial x} + \frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial(x-y)}{\partial x}\right) =$$

$$= \frac{\partial^2 f}{\partial u^2}(x+y,x-y)\cdot 1 + \frac{\partial^2 f}{\partial v\partial u}(x+y,x-y)\,1 - \frac{\partial^2 f}{\partial u\partial v}(x+y,x-y)\cdot 1 - \frac{\partial^2 f}{\partial v^2}(x+y,x-y)\cdot 1$$

3) $g(x,y) = f(x^2+y^2)$ $\qquad f = f(u)$

$$\frac{\partial g}{\partial x}(x,y) = \frac{\partial f}{\partial u}(x^2+y^2)\cdot\frac{\partial(x^2+y^2)}{\partial x} = f'(x^2+y^2)\cdot 2x$$

(1 sčítanec, protože vnější fce má 1 prom.)

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}(x^2+y^2)\cdot\frac{\partial(x^2+y^2)}{\partial y} = f'(x^2+y^2)\cdot 2y$$

$f(x,y) = e^{-(x^2+y^2)}$ ... PD vzhl. k pol. souř.

$x = r\cdot\cos\alpha \qquad y = r\cdot\sin\alpha$

$\tilde{f}(r,\alpha) = e^{-(r^2\cos^2\alpha + r^2\sin^2\alpha)} = e^{-r^2}$

$\frac{\partial\tilde{f}}{\partial r} = -2re^{-r^2} \qquad \frac{\partial\tilde{f}}{\partial\alpha} = 0$ (ihned)

Pomocí ŘP:

$$\frac{\partial\tilde{f}}{\partial\alpha} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial\alpha} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial\alpha} =$$

$$= -2x\,e^{-(x^2+y^2)}\cdot(-r\cdot\sin\alpha) +$$

$$+ \left(-2y\,e^{-(x^2+y^2)}\cdot r\cos\alpha\right) =$$

$$= e^{-r^2}\left(2r^2\cos\alpha\sin\alpha - 2r^2\sin\alpha\cdot\cos\alpha\right) = 0$$